Stochastic Navier–Stokes equations via convex integration

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Introduction

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Consider the Navier-Stokes equation on $\mathbb{T}^3$:

$$\begin{aligned} \partial_t u + u \cdot \nabla u =& \nu \Delta u - \nabla p + \xi, \quad \mathrm{div} u = 0 \\ u(0) =& u_0 \end{aligned} \tag{1}$$

- $u(t, x) \in \mathbb{R}^3$: the velocity field at time t and position x,
- $p(t, x)$: the pressure,
- $\nu > 0$: the viscosity constant
- ξ: trace-class noise

Derivation of Navier-Stokes system: Newton's law

Suppose $u = u(t, x(t))$ and ρ: the density

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \underbrace{\partial_t u}_{\text{variation}} + \underbrace{u \cdot \nabla u}_{\text{convection}} = \underbrace{\nu \Delta u}_{\text{Diffusion}} - \underbrace{\nabla p}_{\text{Internal source}} + \underbrace{f}_{\text{External source}},$$

$$\underbrace{\partial_t \rho + \nabla \cdot (\rho u) = 0}_{\text{mass conservation}} \Rightarrow^{\text{if } \rho = \text{constant}} \mathrm{div} u = 0$$

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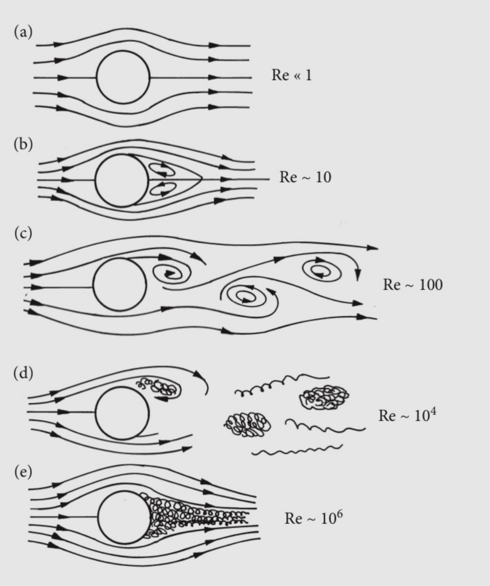
Motivation of Gaussian noise

- stochastic reduction
- regularization by noise
- turbulence **Kolmogorov ('41) theory**
  Zeroth law of turbulence: the inviscid limit $\nu \to 0$

  $$\varepsilon = \liminf_{\nu \to 0} \left(\nu \langle |\nabla u_\nu|^2 \rangle \right) > 0,$$

  $\langle \cdot \rangle$: integration w.r.t. spatial variable and stationary measure μ^ν
  Kolmogorov ('41) theory: 2/3 law, 4/5 law.



$$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p + \xi, \quad \mathrm{Re} \sim 1/\nu$$

Introduction

Deterministic: [Leray34], [Kato, Fujita62], [Temam84], [Constantin, Foias88] [Cafarelli,Kohn, Nirenberg84], [Fefferman 00], [Koch, Tataru01], [Zhang],...

- The global existence of **weak** solutions has been obtained in all dimensions.
- Existence and smoothness of solutions in the three dimensional case remains open (the Millennium Prize problem)./ Small initial data
- [Buckmaster, Vicol AOM19]: Non-uniqueness of analytic weak solutions
- [Albritton, E. Brué, M. Colombo. AOM22] non-uniqueness of Leray solutions for some force

Naiver-Stokes equations driven by trace-class noise

Different notions of solutions

- Martingale solutions/Probabilistically weak solutions: probability measure on the canonical space $C([0,\infty):H^{-3})$
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Known result:

- Leray martingale Markov solutions to stochastic 3D Navier-Stokes have been constructed [Flandoli, Romito PTRF08]

Convex integration for N-S

- Convex integration: iteration procedure a pair $(v_q, \mathring{R}_q)$ is constructed solving the following system

$$\partial_t v_q - \Delta v_q + \mathrm{div}(v_q \otimes v_q) + \nabla p_q = \mathrm{div}\mathring{R}_q \quad \mathrm{div} v_q = 0.$$

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- Key step: Let $w_{q+1} = v_{q+1} - v_q$, then we have

$$\begin{aligned} \mathrm{div} \mathring{R}_{q+1} = & \underbrace{-\Delta w_{q+1} + \partial_t w_{q+1} + \mathrm{div}(v_q \otimes w_{q+1} + w_{q+1} \otimes v_q)}_{\text{linear error}} \\ & + \underbrace{\mathrm{div}\left(w_{q+1} \otimes w_{q+1} + \mathring{R}_q\right)}_{\text{oscillation error: cancelation}} + \nabla p_{q+1} - \nabla p_q. \end{aligned}$$

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$$+ \underbrace{\mathrm{div}\left(w_{q+1} \otimes w_{q+1} + \mathring{R}_q\right)}_{\text{oscillation error: cancelation}} + \nabla p_{q+1} - \nabla p_q.$$

$w_{q+1} \sim \sum_\xi a_\xi(\mathring{R}_q) W_\xi$, where W_ξ is intermittent jets/Beltrami waves/Mikado waves.

- The space concentration ensure the linear error is small in L^1
- $\int W_\xi \otimes W_\xi \simeq 1$ and $a_\xi(\mathring{R}_q) \approx \sqrt{-\mathring{R}_q}$ oscillates slowly

Nonuniqueness in law for Stochastic 3D Navier-Stokes/Euler equations

Problem 1: Non-uniqueness in law hold?

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1. Regularization by noise:
   - The ODE
     $$dX_t = |X_t|^\alpha dt, \quad X_0 = 0, \quad \alpha \in (0,1)$$
     has infinitely many solutions
   - The SDE
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2. Example:
   $$dX_t = sign(X_t) dB_t$$
   - Pathwise uniqueness does not hold: If X is a solution then $-X$ is also a solution.
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Theorem (Hofmanová, Zhu, Z. 19/ CPAM22)

Non-uniqueness in law holds for the stochastic 3D Navier- Stokes/ Euler system.

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Theorem (Hofmanová, Zhu, Z. AOP22+)

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- [Chen, Dong, Z. 22] Sharp nonuniqueness and global probabilistically strong solutions in higher dimensions/ Euler equations
- [Lü, Z. 22] Global probabilistically strong solutions for power law equations

Ergodic hypothesis

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Known result:

- 2d Navier-Stokes: Uniqueness of invariant measure [Hairer, Mattingly AOM06]
- 3d Navier-Stokes with non-degenerate noise: Every Markov selection has a unique invariant measure [Flandoli, Romito PTRF08]

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There exist

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to the stochastic 3D Navier–Stokes and Euler equations.

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For any $\nu_n \to 0$, $\exists$ stationary solutions u_n to SNS with $\nu = \nu_n$ so that $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}; L^2_\sigma)$ and every accumulation point is a stationary solution to the stochastic Euler equations.

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Idea of proof: Stochastic convex integration

$$\sup_{\nu} \sup_{s\in\mathbb{R}} \Big(\mathbf{E} \sup_{s\le t\le s+1} \|u(t)\|^2_{H^\vartheta} + \mathbf{E}\|u(t)\|^2_{C^\vartheta([s,s+1];L^2)}\Big) < \infty.$$

Anomalous dissipation

Theorem (Hofmanová, Zhu, Z. 22)

For any $\epsilon > 0$, $\exists\ \nu_n \to 0$ *and stationary processes* $(u_n, \mathring{R}_n) \in C(\mathbb{R}; H^1) \times C(\mathbb{R}; L^1)$ *satisfying the following stochastic Navier–Stokes–Reynolds equations*

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$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{0\le s\le 1} \|\mathring{R}_n(s)\|_{L^1}\right] = 0,$$

and

$$\liminf_{n\to\infty} \nu_n \mathbf{E}\|\nabla u_n\|_{L^2}^2 \ge \epsilon + \frac{1}{2}\mathrm{tr}(GG^*).$$

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$$\liminf_{n\to\infty} \nu_n \mathbf{E}\|\nabla u_n\|_{L^2}^2 \ge \epsilon + \frac{1}{2}\mathrm{tr}(GG^*).$$

Furthermore, the laws $\mathcal{L}[u_n]$, $n \in \mathbb{N}$, *is tight in* $C(\mathbb{R}; L^2_\sigma)$ *and every accumulation point is a stationary solution to the stochastic Euler equations.*

see [Brue, De Lellis 22]

Thank you !